So

$$\sin 3\theta = \frac{KL}{AD} = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}.$$

It is well-known that $\sin 54^\circ = \dfrac{\sqrt{5}+1}{4}$ (*Editor's comment : this can be derived by elementary procedure and several solvers provided a detailed proof for this.*) Since

$$\angle A < \frac{\pi}{2} \quad \text{and} \quad \sin\angle A = \sin(\pi - 3\theta) = \sin 3\theta = \frac{\sqrt{5}+1}{4},$$

we conclude that $\angle A = 54^\circ = \dfrac{3\pi}{10}$, from which it follows that

$$\angle C = \theta = \frac{1}{3}(\pi - \angle A) = \frac{1}{3}\left(\pi - \frac{3\pi}{10}\right) = \frac{7\pi}{30} = 42^\circ$$

and $\angle B = 2\theta = \dfrac{7\pi}{15} = 84^\circ$.

3829. *Proposed by Michel Bataille.*

Let a, b, c be positive real numbers and $\Delta = a^2+b^2+c^2-(ab+bc+ca)$. Improve the well known inequality $\Delta \geq 0$ by proving that

$$\Delta \geq \left(\frac{a(a-b)^2(a-c)^2+b(b-c)^2(b-a)^2+c(c-a)^2(c-b)^2}{a+b+c}\right)^{\frac{1}{2}}.$$

Solved by A. Alt ; AN-anduud Problem Solving Group ; Ş. Arslanagić ; R. Barbara ; D. Bailey, E. Campbell and C. Diminnie ; M. Dincă ; N. Evgenidis ; O. Kouba ; D. Koukakis ; K. -W. Lau ; S. Malikić ; P. McCartney ; C.R. Pranesachar ; D. Smith ; T. Zvonaru and N. Stanciu ; and the proposer. There was one flawed solution. The more efficient approaches are summarized below.

Preliminaries. We establish notation and basic facts. The summation sign will refer to cyclic sums :

$$\sum f(a,b,c) = f(a,b,c)+f(b,c,a)+f(c,a,b).$$

$$\begin{aligned}\Delta &= a^2+b^2+c^2-ab-bc-ca\\ &= (a-b)(a-c)+(b-c)(b-a)+(c-a)(c-b)\\ &= (a-b)(a-c)+(b-c)^2 = (b-c)(b-a)+(c-a)^2 = (c-a)(c-b)+(a-b)^2\\ &= \frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \geq 0.\end{aligned}$$

Then we have :

$$\begin{aligned} A &= a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \\ &= \sum a^3 - \sum(a^2b + ab^2) + 3abc. \end{aligned}$$

$$\begin{aligned} B &= a(a-b)^2(a-c)^2 + b(b-a)^2(b-c)^2 + c(c-a)^2(c-b)^2 \\ &= \sum a^5 + \sum(a^3b^2 + a^2b^3) + 4\sum a^3bc - 3\sum ab^2c^2 - 2\sum(a^4b + ab^4) \\ &= \Delta A. \end{aligned}$$

Finally,

$$\begin{aligned} \Gamma = (a+b+c)\Delta^2 - B = \Delta[\Delta(a+b+c) - A] &= \Delta\left[\sum(a^2b + ab^2) - 6abc\right] \\ &= \Delta[(a+b)(b+c)(c+a) - 8abc]. \end{aligned}$$

The problem requires it to be shown that $\Gamma \geq 0$. Equality will occur if and only if $a = b = c$.

Solution 1, by Ş. Arslanagić; Kee-Wai Lau; Salem Malikić; and Phil McCartney (all independently).

$$\Gamma = \Delta(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 - 6abc) \geq 0,$$

by the arithmetic-geometric means inequality.

Edtor's comment. E. Nikolaos used the fact that $\Gamma = \Delta[(a+b)(b+c)(c+a) - 8abc]$ and noted that $a+b \geq 2\sqrt{ab},\ b+c \geq 2\sqrt{bc},\ c+a \geq 2\sqrt{ca}$.

Solution 2, by Titu Zvonaru and Neculai Stanciu.

$$\begin{aligned} \Gamma &= a[\Delta - (a^2 - ab - ac + bc)][\Delta + (a^2 - ab - ac + bc)] \\ &\qquad + b[\Delta - (b^2 - bc - ba + ca)][\Delta + (b^2 - bc - ba + ca)] \\ &\qquad + c[\Delta - (c^2 - ca - ca + ab][\Delta + (c^2 - ca - cb + ab)] \\ &= (ab^4 + a^4b + bc^4 + b^4c + ca^4 + c^4a) + 6(ab^2c^2 + a^2bc^2 + a^2b^2c) \\ &\qquad - 8(a^3bc + ab^3c + abc^3) \\ &= a(b-c)^4 + b(c-a)^4 + c(a-b)^4 \geq 0. \end{aligned}$$

Solution 3, by the AN-anduud Problem Solving Group; and Dimitrios Koukakis (independently).

Observe that

$$(uv + vw + wu)^2 = u^2v^2 + v^2w^2 + w^2u^2$$

when $u + v + w = 0$. Therefore, setting $(u, v, w) = (a-b, b-c, c-a)$, we obtain

$$\begin{aligned} (a+b+c)\Delta^2 &= (a+b+c)[(a-b)^2(a-c)^2 + (b-c)^2(b-a)^2 + (c-a)^2(c-b)^2] \\ &\geq a(a-b)^2(a-c)^2 + b(b-c)^2(b-a)^2 + c(c-a)^2(c-b)^2 = B. \end{aligned}$$

Solution 4, by Omran Kouba, modified by the editor.

Without loss of generality, assume that $a \geq b \geq c$. Then

$$\Delta \geq (a-b)(a-c) \geq |b-a|(b-c);$$
$$\Delta \geq (c-a)(c-b) = (a-c)(b-c).$$

Then Δ^2 is not less than each of $(a-b)^2(a-c)^2$, $(b-a)^2(c-a)^2$ and $(c-a)^2(c-b)^2$. Therefore Δ^2 is not less than the weighted average $B/(a+b+c)$ of these terms.

Solution 5, by Dionne Bailey, Elsie Campbell and Charles Diminnie.

Since $2\Delta = (a-b)^2 + (b-c)^2 + (c-a)^2$, then

$$4\Delta^2 = 4(a-b)^2(c-a)^2 + [(a-b)^2-(c-a)^2]^2 + (b-c)^4 + 2(a-b)^2(b-c)^2 + 2(b-c)^2(c-a)^2,$$

so that $\Delta^2 \geq (a-b)^2(a-c)^2$.

Similarly, $\Delta^2 \geq (b-c)^2(b-a)^2$ and $\Delta^2 \geq (c-a)^2(c-b)^2$. Hence the right side of the inequality does not exceed $(a+b+c)^{-1/2}(a\Delta^2 + b\Delta^2 + c\Delta^2)^{1/2} = \Delta$.

Solution 6, by Arkady Alt.

Note that

$$\begin{aligned} B &= \sum a(a-b)(a-c)[\Delta - (b-c)^2] \\ &= \Delta \sum a(a-b)(a-c) + (a-b)(b-c)(c-a)\sum a(b-c) \\ &= \Delta \sum a[\Delta - (b-c)^2] + 0 = \Delta^2(a+b+c) - \Delta\sum a(b-c)^2. \end{aligned}$$

Hence

$$\Gamma = \Delta \sum a(b-c)^2 \geq 0.$$

Solution 7, by C.R. Pranesachar.

$$\begin{aligned} \Gamma &= (a+b+c)\Delta^2 - B = \Delta[(a+b+c)B - A] \\ &= (b+c)(a-b)^2(a-c)^2 + (a+c)(b-a)^2(b-c)^2 + (a+b)(c-a)^2(c-b)^2 \geq 0. \end{aligned}$$

3830. *Proposed by Tigran Hakobyan.*

Let $a > 0$. Define the sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers by

$$a_1 = a, a_{n+1} = a_n + \{a_n\}, n \geq 1$$

where $\{x\}$ is the fractional part of x. Find all $a > 0$ such that the sequence $\{a_n\}_{n=0}^{\infty}$ defined above is bounded.

Solved by A. Alt; R. Barbara; O. Kouba; K. Lewis; P. Perfetti; D. Stone and J. Hawkins; D. Văcaru; and the proposer. Two incorrect solutions were received. We present a composite of solutions by the listed solvers.